INDIAN STATISTICAL INSTITUTE, BANGALORE

Mid-semestral Examination 2025–26 (First Semester)

M. Math. 1st Year **Algebra I**

Date: 10.09.2025 Marks: 100 Duration: 2 Hours

Instructions:

- Attemt **any five** of the following problems. Justify all your steps clearly. Feel free to use any result proved in class unless you have been asked to prove the same.
- R denotes a commutative ring with unity. All other notations are standard unless specified otherwise.

1. (a) Let $K_1, \ldots, K_n$ be fields. Prove that the product ring $K_1 \times \cdots \times K_n$ has exactly n prime ideals, each of which is maximal. Describe them explicitly. [10+10]

   (b) Examine whether
   $$\mathbb{C}[X,Y]/(XY) \cong \mathbb{C}[X,Y]/(X) \times \mathbb{C}[X,Y]/(Y)$$
   as $\mathbb{C}$-algebras.

2. (a) Prove that $R[X]$ has infinitely many prime ideals. [10+10]

   (b) Compute $\mathbb{Z}[i]/(1+4i)$.

3. (a) For a UFD R, show that the following are equivalent. [10+10]

   i. R is a PID.

   ii. Any two non-associate primes generate the unit ideal.

   (b) Show that if $\mathfrak{m}$ is a maximal ideal of R, then any prime ideal of $R[X]$ strictly containing $\mathfrak{m}[X]$ must be maximal.

4. (a) Show that if R is an infinite commutative ring with unity and has only finitely many units, then R has infinitely many maximal ideals. [10+10]

   (b) Let k be a field. Show that $k[X, 1/X]$ is a Euclidean domain by defining suitable Euclidean function.

5. (a) Examine whether the following rings are isomorphic. [10+10]

   i. $\mathbb{C}[X,Y]/(X-Y^2)$ and $\mathbb{C}[X,Y]/(XY-1)$

   ii. $\mathbb{C}[X,Y]/(XY-1)$ and $\mathbb{C}[X,Y]/(X^2+Y^2-1)$

   (b) Construct a localization of $\mathbb{Z}[X]$ with exactly two maximal ideals. Justify your claim.

6. (a) Let R be an integral domain and let $a, b \in R$ such that $Ra + Rb = R$. Show that there is an isomorphism $R[X]/(aX-b) \cong R[1/a]$. [12+8]

   (b) State, with brief justification, which of the following are local rings.
   (i) $\mathbb{Q}[X]/(X^9)$, (ii) $\mathbb{C}[X]/(X^2+1)$, (iii) $\mathbb{R}[X,Y]/(X^2, XY, Y^2)$,(iv) $\mathbb{Z}[\frac{1}{3}, \frac{1}{5}, \cdots, \frac{1}{p}, \cdots]$.